9.1 Periodic Functions and Fourier Series

revisit Taylor series → break $f(x)$ into its building blocks x^n

for example, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

→ to build e^x using x^n, we need one part of 1, one part of x, $\frac{1}{2!}$ parts of x^n, ...

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

to have a Taylor series, $f(x)$ needs to be infinitely differentiable at $x = a$

these x^n are called <u>basis functions</u>

(~~exactly~~ like $\vec{i}$, $\vec{j}$, $\vec{k}$ we use to build any $\mathbb{R}^3$ vector)

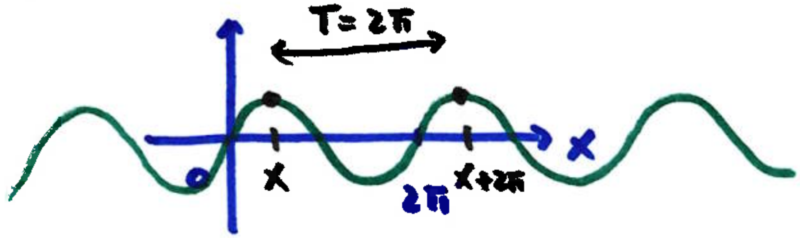
Fourier series is very similar: instead of x^n, the basis functions are now $\cos(nx)$ and $\sin(nx)$

to have a Fourier Series, $f(x)$ need to be periodic and piecewise ~~continuous~~ smooth

a function $f(x)$ is periodic w/ period T if

$$f(x) = f(x+T)$$

for example, $\sin(x) = \sin(x+2\pi)$



$$\sin(x) = \sin(x+2\pi) = \sin(x+4\pi) = \sin(x+6\pi)$$

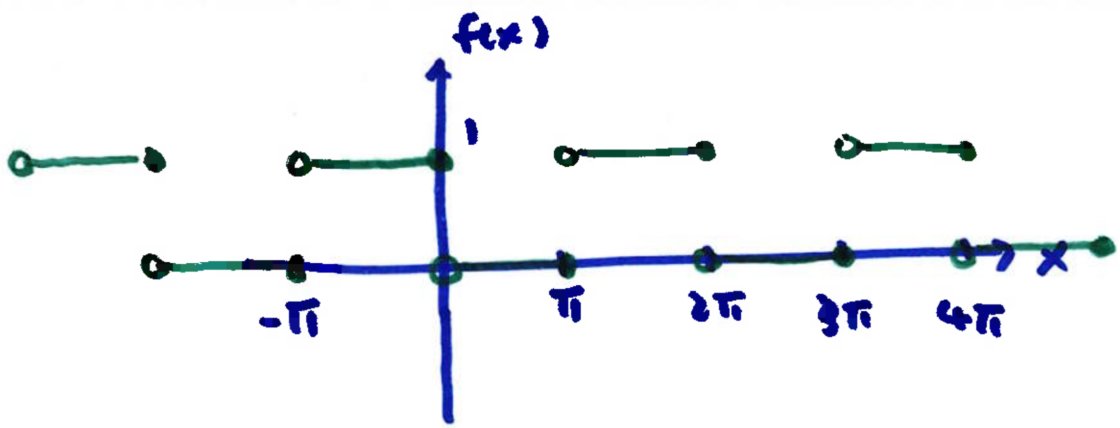
$$= \dots = \sin(x + k\cdot 2\pi) \quad k = 1, 2, 3, \dots$$

$f(x) = f(x+T) = f(x+k\cdot T)$ any integer multiple of period is also a period the shortest period T is the <u>fundamental period</u>

what is T for $\sin(3x)$? $\quad T = \frac{2\pi}{3}$

this is a periodic function that is piecewise ~~continuous~~ smooth

period 2π



$$f(x) = \begin{cases} 1 & -\pi < x \leq 0 \\ 0 & 0 < x \leq \pi \end{cases} \quad \text{period } 2\pi$$

(specify two half periods then state period)

this clunky function has a Fourier series (good because sines and cosines are easy to work with)

Fourier series of $f(x)$ with period $\underline{2\pi}$ defined on $\underline{-\pi < x < \pi}$ is

$a_0 = a_0 \cdot 1 = a_0 \cdot \cos(0x)$

$$f(x) = \frac{1}{2}a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots$$
$$+ b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots$$
$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

(as a comparison, Taylor series is $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

how to find a_n and b_n ?

Some important properties of cosines and sines

$$\int_{-\pi}^{\pi} \cos(\alpha x)\cos(\beta x)\,dx = \begin{cases} \pi & \text{if } \alpha=\beta\neq 0 \\ 2\pi & \text{if } \alpha=\beta=0 \\ 0 & \text{if } \alpha\neq\beta \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(\alpha x)\sin(\beta x)\,dx = \begin{cases} \pi & \text{if } \alpha=\beta\neq 0 \\ 0 & \text{if } \alpha\neq\beta \text{ or } \alpha=\beta=0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(\alpha x)\sin(\beta x)\,dx = 0 \quad \text{for all } \alpha, \beta$$

Cosines and sines are <u>mutually orthogonal</u>

$$f(x) = \tfrac{1}{2}a_0 + a_1\cos(x) + a_2\cos(2x) + \dots + b_1\sin(x) + b_2\sin(2x) + \dots$$

multiply both sides by $\cos(nx)$ and integrate over $-\pi < x < \pi$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \int_{-\pi}^{\pi} \tfrac{1}{2}a_0 \cdot \cos(0x)\cos(nx)\,dx$$

go to 0 if $n \neq 0, 1, 2, \text{or} \dots$

$$+ \int_{-\pi}^{\pi} a_1\cos(x)\cos(nx)\,dx + \int_{-\pi}^{\pi} a_2\cos(2x)\cos(nx)\,dx$$

$$+ \dots + \cancelto{0}{\int_{-\pi}^{\pi} b_1\sin(x)\cos(nx)\,dx} + \cancelto{0}{\int_{-\pi}^{\pi} b_2\sin(2x)\cos(nx)\,dx}$$

$$+ \cancelto{0}{\dots}$$

$$= \int_{-\pi}^{\pi} a_n\cos(nx)\cos(nx)\,dx = a_n \cdot \pi$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx} \qquad n = 0, 1, 2, 3, \ldots$$

likewise, if we multiply by $\sin(nx)$ and integrate over $-\pi < x < \pi$, we get

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx} \qquad n = 1, 2, 3, \ldots$$

let's try it on $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } 0 < x < \pi \end{cases}$ period 2π

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 1 \cdot dx = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \cos(nx)\,dx = \frac{1}{n\pi}\sin(nx)\Big|_{-\pi}^{0} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)\,dx$$

$$= -\frac{1}{n\pi}\cos(nx)\Big|_{-\pi}^{0} = -\frac{1}{n\pi}\left(1 - \cos(n\pi)\right)$$

$\downarrow$

-1 if n is odd

1 if n is even

$\Rightarrow (-1)^n$

$$b_n = -\frac{1}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$